Problem 91

Distances between points in a plane do not change when a coordinate system is rotated. In other words, the magnitude of a vector is *invariant* under rotations of the coordinate system. Suppose a coordinate system S is rotated about its origin by angle φ to become a new coordinate system S$'$, as shown in the following figure. A point in a plane has coordinates (x, y) in S and coordinates (x', y') in S$'$.

(a) Show that, during the transformation of rotation, the coordinates in S$'$ are expressed in terms of the coordinates in S by the following relations:

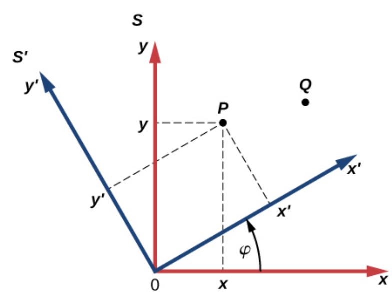
$$\begin{cases} x' = x\cos\varphi + y\sin\varphi \\ y' = -x\sin\varphi + y\cos\varphi \end{cases}.$$

(b) Show that the distance of point P to the origin is invariant under rotations of the coordinate system. Here, you have to show that

$$\sqrt{x^2 + y^2} = \sqrt{x'^2 + y'^2}.$$

(c) Show that the distance between points P and Q is invariant under rotations of the coordinate system. Here, you have to show that

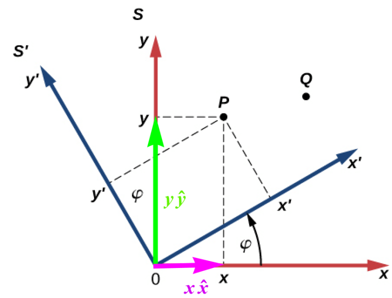
$$\sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2} = \sqrt{(x'_P - x'_Q)^2 + (y'_P - y'_Q)^2}.$$



Solution

Part (a)

Let $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{x}}'$, and $\hat{\mathbf{y}}'$ be the unit vectors along the x-axis, y-axis, x'-axis, and y'-axis, respectively. For the point P given by (x, y), consider the vectors along the x- and y-axes.



To get x', take the components of these vectors along the x'-axis and add them together.

$$\begin{aligned} x' &= (x\hat{\mathbf{x}}) \cdot \hat{\mathbf{x}}' + (y\hat{\mathbf{y}}) \cdot \hat{\mathbf{x}}' \\ &= x(\hat{\mathbf{x}} \cdot \hat{\mathbf{x}}') + y(\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}') \\ &= x|\hat{\mathbf{x}}||\hat{\mathbf{x}}'| \cos\varphi + y|\hat{\mathbf{y}}||\hat{\mathbf{x}}'| \cos(90^\circ - \varphi) \\ &= x(1)(1)\cos\varphi + y(1)(1)\sin\varphi \\ &= x\cos\varphi + y\sin\varphi \end{aligned}$$

To get y', take the components of these vectors along the y'-axis and add them together.

$$\begin{aligned} y' &= (x\hat{\mathbf{x}}) \cdot \hat{\mathbf{y}}' + (y\hat{\mathbf{y}}) \cdot \hat{\mathbf{y}}' \\ &= x(\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}') + y(\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}') \\ &= x|\hat{\mathbf{x}}||\hat{\mathbf{y}}'| \cos(90^\circ + \varphi) + y|\hat{\mathbf{y}}||\hat{\mathbf{y}}'| \cos\varphi \\ &= x(1)(1)(-\sin\varphi) + y(1)(1)\cos\varphi \\ &= -x\sin\varphi + y\cos\varphi \end{aligned}$$

Part (b)

The distance from the origin to point P is invariant because

$$\begin{aligned} \sqrt{x'^2 + y'^2} &= \sqrt{(x\cos\varphi + y\sin\varphi)^2 + (-x\sin\varphi + y\cos\varphi)^2} \\ &= \sqrt{(x^2\cos^2\varphi + 2xy\cos\varphi\sin\varphi + y^2\sin^2\varphi) + (x^2\sin^2\varphi - 2xy\sin\varphi\cos\varphi + y^2\cos^2\varphi)} \\ &= \sqrt{x^2(\cos^2\varphi + \sin^2\varphi) + y^2(\sin^2\varphi + \cos^2\varphi)} \\ &= \sqrt{x^2 + y^2}. \end{aligned}$$

Part (c)

Let point P be (x_P, y_P), and let point Q be (x_Q, y_Q).

$$\sqrt{(x'_P - x'_Q)^2 + (y'_P - y'_Q)^2} = \sqrt{[(x_P \cos\varphi + y_P \sin\varphi) - (x_Q \cos\varphi + y_Q \sin\varphi)]^2 + [(-x_P \sin\varphi + y_P \cos\varphi) - (-x_Q \sin\varphi + y_Q \cos\varphi)]^2}$$

$$= \sqrt{[(x_P - x_Q)\cos\varphi + (y_P - y_Q)\sin\varphi]^2 + [(x_Q - x_P)\sin\varphi + (y_P - y_Q)\cos\varphi]^2}$$

Note that

$$[(x_P - x_Q)\cos\varphi + (y_P - y_Q)\sin\varphi]^2 = (x_P - x_Q)^2\cos^2\varphi + 2(x_P - x_Q)(y_P - y_Q)\cos\varphi\sin\varphi + (y_P - y_Q)^2\sin^2\varphi$$

and

$$[(x_Q - x_P)\sin\varphi + (y_P - y_Q)\cos\varphi]^2 = (x_Q - x_P)^2\sin^2\varphi + 2(x_Q - x_P)(y_P - y_Q)\sin\varphi\cos\varphi + (y_P - y_Q)^2\cos^2\varphi$$

$$= (x_P - x_Q)^2\sin^2\varphi - 2(x_P - x_Q)(y_P - y_Q)\cos\varphi\sin\varphi + (y_P - y_Q)^2\cos^2\varphi.$$

Therefore,

$$\sqrt{(x'_P - x'_Q)^2 + (y'_P - y'_Q)^2} = \sqrt{(x_P - x_Q)^2(\cos^2\varphi + \sin^2\varphi) + (y_P - y_Q)^2(\sin^2\varphi + \cos^2\varphi)}$$

$$= \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2},$$

which means the distance between points P and Q is invariant under rotations of the coordinate system.